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# Constrained Hamiltonian Systems and Gauge Theories

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We are concerned with the covariant description of the Hamiltonian formalism for constrained field systems. The relation with the Lagrangian formalism is considered and applications to gauge theories are given. Both formalisms are developed on the same space, namely the momentum space. The equivalence of solutions is shown to hold for affine and quadratic Lagrangians. The Yang–Mills equations are put into a Hamiltonian form by means of a *complete* family of Hamiltonians. This completeness property appears in a nice way as a *gauge-type* condition connected with the Hamilton equations and generalizing the notion of gauge condition usually dealt with in gauge theory.

## 1. INTRODUCTION

As is well known, the geometric arena of classical field theory (and other physical models) is a fibered manifold $E \to M$. The base manifold $M$ may be a space-time manifold, a space of parameters, etc. Classical fields are sections of this fibered manifold.

The Lagrangian formulation of field theory is developed on $J^1E$, the first-order jet prolongation of $E \to M$ (or on higher order prolongations for higher order theories), and the field equations are second-order equations, i.e., Lagrange's equations (Krupka, 1971; Mangiarotti and Modugno, 1983).

In the past decades, the standard Hamiltonian machinery based on Dirac's ideas has been developed (Hanson *et al.*, 1976; Sundermeyer, 1982). This leads to infinite-dimensional symplectic manifolds (Gotay *et al.*, 1978; Kosmann-Schwarzbach, 1981; Bergvelt and De Kerf, 1986). The main goal consists in establishing simultaneous commutation relations for quantum field theories (Sundermeyer, 1982; Faddeev and Slavnov, 1991).

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On the other hand, in more recent years, many people have contributed to setting up a covariant Hamiltonian formalism for field theory, i.e., one which does not necessitate any space-time decomposition (Kijowski, 1973; Günther, 1987; Cariñena *et al.*, 1991; Gotay, 1991; Sardanashvily and Zakharov, 1992a,b). It seems that the choice of the space in which such a Hamiltonian description is formulated is far from being unique.

The relationship between the Lagrangian and Hamiltonian formalisms has also been extensively studied, both in mechanics (Batlle *et al.*, 1986) and field theory (Gotay, 1991; Goldschmidt and Stenberg, 1973; Dedecker, 1977; Krupka, 1982), generally under strong regularity conditions.

However, due to the presence of gauge invariance, the interesting physical models never have this kind of regularity. Nevertheless they still retain a weaker regularity property (in the sequel we shall refer to these systems as almost regular, a.r.). For them, the problem of the equivalence between the Lagrangian and Hamiltonian approaches is more subtle and deserves further investigation.

In this paper we study this problem for first-order field theories (similar questions for higher order theories are under study). We do not follow the traditional approach in which the Lagrangian and Hamiltonian formalisms are set up on different spaces. Instead, following Kijowski and Tulczyjew (1979), we develop both of them on $P = \wedge^{m-1} T^*M \otimes V^*E$. In particular, we regard the Lagrange and Hamilton field equations as partial differential equations (pde's) for sections of the composite fibered manifold $P \to E \to M$.

The main results and the organization of the paper are as follows. In Section 2 we fix our notation, summarizing some basic facts about jet manifolds, the momentum manifold $P$, and its related structures.

In Section 3 we study the Lagrangian and Hamiltonian formalisms independently of each other but in a rather 'symmetric' way. The solutions of the Lagrange equations are characterized by an affine subbundle $\mathbf{L} \subset J^1P$ with base manifold $J^1E$. We also introduce the concepts of Hamiltonian form and Hamiltonian map (Zakharov, 1992; Sardanashvily and Zakharov, 1992a,b; Giachetta *et al.*, 1993). These are basic concepts for our considerations.

Section 4 is the main section. Here we study a.r. Lagrangian densities. Using the family of Hamiltonian forms parametrized by all the Hamiltonian maps, we show that the Lagrange equations are equivalent to the corresponding family of Hamilton equations. The constrained Hamilton equations are regarded as an affine subbundle $\mathbf{H} \subset J^1P$ with base manifold $Q \subset P$, the constraint manifold. It turns out that $\mathbf{L} \subset \mathbf{H}$ and there are systems for which $\mathbf{L} \neq \mathbf{H}$. This means that, in general, the Lagrange and the constrained Hamilton equations are not equivalent. However, if the Lagrangian density is affine or quadratic in the field derivatives (as is the case for most physical models), then we show that $\mathbf{L} = \mathbf{H}$.

In Section 5 we consider the (free) Yang–Mills Lagrangian density. Now the configuration bundle $E \equiv C \to M$ is a bundle of principal connections, which has some specific properties. In particular, both the first jet manifold $J^1C$ and the momentum manifold $P$ admit a canonical splitting over $C$. A consequence of this geometrical feature is that there is a natural family of Hamiltonian maps, and hence of Hamiltonian forms. It turns out that this family is *complete* in the sense that it is the minimal family whose corresponding Hamilton equations are equivalent to the Yang–Mills equations. This completeness property appears in a nice way as a kind of *gauge-type* condition connected with the Hamilton equations.

## 2. THE MOMENTUM SPACE

All manifolds and maps throughout the paper will be smooth ($C^\infty$). For more details on the jet formalism we refer the reader to Mangiarotti and Modugno (1982) and Saunders (1989).

### 2.1. Jet Manifolds and Connections

Let $M$ be a manifold of dimension $m \geq 1$ with local coordinates $(x^\lambda)$, $1 \leq \lambda \leq m$. We denote by $TM$ and $T^*M$ the tangent and cotangent spaces of $M$ and use the symbols $\otimes$, $\vee$, and $\wedge$ for tensor, symmetric, and exterior products, respectively.

Let $E \to M$ be a fibered manifold of dimension $m + l$ ($E$ is the total space and $M$ is the base) with fiber coordinates $(x^\lambda, y^i)$, $1 \leq i \leq l$. We denote by $VE \subset TE$ the vertical subspace of $TE$, with local coordinates $(x^\lambda, y^i, \dot{y}^i)$, and by $V^*E$ its dual (over $E$).

We have the tower of fibered manifolds

$$J^1E \to E \to M \tag{2.1}$$

where $J^1E$ is the first jet manifold with fiber coordinates $(x^\lambda, y^i, y^i_\lambda)$. If we consider a (local) section of $E \to M$, i.e.,

$$s\colon\ M \to E, \qquad (x^\lambda, y^i) \circ s = (x^\lambda, s^i)$$

where $s^i$ are (local) functions on $M$, then the (local) expression of its first jet prolongation $j^1s\colon M \to J^1E$ is given by $(x^\lambda, y^i, y^i_\lambda) \circ j^1s = (x^\lambda, s^i, \partial s^i/\partial x^\lambda)$.

In the sequel, we shall denote by $\partial_\lambda \equiv \partial/\partial x^\lambda$, $\partial_i \equiv \partial/\partial y^i$, and $\partial_i^\lambda \equiv \partial/\partial y^i_\lambda$ the coordinate fields associated with $x^\lambda$, $y^i$, and $y^i_\lambda$, respectively.

A basic fact concerning $J^1E$ is the existence of a canonical injection over $E$,

$$J^1E \overset{\lambda}{\hookrightarrow} T^*M \otimes TE, \qquad \lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i) \tag{2.2}$$

from which it follows that $J^1E \to E$ is an affine bundle whose associated vector bundle is $T^*M \otimes VE \to E$. As a rule, induced linear fiber coordinates on this bundle will be denoted by $(x^\lambda, y^i, \bar{y}^i_\lambda)$.

### 2.2. The Structure of the Momentum Space

The fibered manifold $E \to M$ is the *configuration manifold* of a given field system, while

$$P = \overset{m-1}{\bigwedge} T^*M \otimes V^*E \to E \to M \tag{2.3}$$

is the corresponding *momentum manifold*. Obviously $P \to E$ is a vector bundle. Induced fiber coordinates on $P$ and its first jet manifold are denoted by $(x^\lambda, y^i, p^\mu_i)$ and $(x^\lambda, y^i, p^\mu_i, y^i_\lambda, p^\mu_{\lambda i})$, respectively. Note that $J^1P \to J^1E$ is a vector bundle, but $J^1P \to P$ is an affine bundle whose associated vector bundle is $T^*M \otimes VP \to P$. The induced linear fiber coordinates on this bundle are denoted by $(x^\lambda, y^i, p^\mu_i, \bar{y}^i_\lambda, \bar{p}^\mu_{\lambda i})$.

As happens for the cotangent space of a manifold, the momentum space (2.3) carries a canonical *vector-valued Liouville form* $\theta$. More precisely, $\theta$ is the base-tangent $(m + 1)$-form on $P$ given by the canonical injection

$$P \overset{\theta}{\hookrightarrow} \overset{m+1}{\bigwedge} T^*E \otimes TM$$

$$\theta = p^\lambda_i dy^i \wedge \omega \otimes \partial_\lambda, \qquad \omega = dx^1 \wedge \cdots \wedge dx^m$$

Contracting $\theta$ with the canonical injection (2.2), we get the following scalar form:

$$\Theta = -\lambda \rfloor \theta: \quad P \underset{E}{\times} J^1E \to \overset{m}{\bigwedge} T^*E$$

$$\Theta = p^\lambda_i\, \theta^i \wedge \omega_\lambda, \qquad \theta^i = dy^i - y^i_\mu dx^\mu \tag{2.4}$$

There are two basic objects related to the form (2.4). The first is

$$\Omega_L = -d_h\Theta: \quad J^1P \to \overset{m}{\bigwedge} T^*M \otimes V^*J^1E$$

$$\Omega_L = (p^\lambda_{\lambda i} dy^i + p^\lambda_i dy^i_\lambda) \otimes \omega \tag{2.5}$$

where $d_h$ denotes the horizontal exterior derivative on jet manifolds (Saunders, 1989). This form will be used in the Lagrangian formalism. Note that $\Omega_L$ can be seen as a linear epimorphism over $J^1E$.

The second object related to (2.4) is

$$\Omega_H = d_v\Theta\colon \quad J^1P \to \bigwedge^{m+1} T^*P$$

$$\Omega_H = dp_i^\lambda \wedge dy^i \wedge \omega_\lambda - y_\lambda^i dp_i^\lambda \wedge \omega + p_{\lambda i}^\lambda dy^i \wedge \omega \tag{2.6}$$

where $d_v$ denotes the vertical exterior derivative on jet manifolds (Saunders, 1989). This form will be used in the Hamiltonian formalism. Note that $\Omega_H$ can be seen as an affine morphism over $P$ whose associated linear map is

$$T^*M \otimes VP \xrightarrow{\bar{\Omega}_H} \bigwedge^{m} T^*M \otimes V^*P \subset \bigwedge^{m+1} T^*P$$

$$\overline{\Omega}_H = -\overline{y}_\lambda^i dp_i^\lambda \wedge \omega + \overline{p}_{\lambda i}^\lambda dy^i \wedge \omega \tag{2.7}$$

Since $\bar{\Omega}_H$ is an epimorphism, it follows that the bundle $\operatorname{Im} \Omega_H \to P$ is an affine subbundle of $\wedge^{m+1} T^*P \to P$ whose associated vector bundle is $\wedge^m T^*M \otimes V^*P \to P$.

The following commutative diagram summarizes the discussion:

$$\begin{tikzcd}
\bigwedge^{m} T^*M \otimes V^*J^1E \arrow[dr] & & J^1P \arrow[ll, "\Omega_L"'] \arrow[rr, "\Omega_H"] \arrow[dl] \arrow[dr] & & \operatorname{Im} \Omega_H \subset \bigwedge^{m+1} T^*P \arrow[dl] \\
& J^1E \arrow[dr] & & P \arrow[dl] & \\
& & E \arrow[d] & & \\
& & M & &
\end{tikzcd}$$

Its left and right parts are the geometric arenas to set up the Lagrangian and Hamiltonian formalisms, respectively. From the diagram, we see that with the help of suitable morphisms $J^1E \to P$ and $P \to J^1E$ (over $E$), we can hope to relate the two formalisms. The morphisms $J^1E \to P$ are the *Legendre maps*, while the morphisms $P \to J^1E$ are the *Hamiltonian maps*.

## 3. LAGRANGIAN AND HAMILTONIAN FORMALISMS

### 3.1. Lagrangian Formalism

As is well known, the basic concept of the Lagrangian formalism is that of the Lagrangian density. A (first order) *Lagrangian density* is a form

$$\mathcal{L}\colon \quad J^1E \to \bigwedge^{m} T^*M, \qquad \mathcal{L} = L\omega, \quad \omega = dx^1 \wedge \cdots \wedge dx^m \tag{3.1}$$

where $L$ is a local function on $J^1E$.

A *Legendre map* is a morphism over $E$, i.e.

$$\Pi: \quad J^1E \to P, \qquad (x^\lambda, y^i, p_i^\lambda) \circ \Pi = (x^\lambda, y^i, \Pi_i^\lambda) \tag{3.2}$$

where $\Pi_i^\lambda$ are local functions on $J^1E$. Given a Lagrangian density $\mathcal{L}$, there is a Legendre map canonically associated with it, namely

$$\Pi: \quad J^1E \to P, \qquad \Pi_i^\lambda = \partial_i^\lambda L \tag{3.3}$$

The *Poincaré–Cartan form* associated with $\mathcal{L}$ is (Mangiarotti and Modugno, 1983)

$$\mathcal{H}_{\mathcal{L}} = (\Theta + \mathcal{L}) \circ \Pi: \quad J^1E \to \overset{m}{\wedge} T^*E$$

$$\mathcal{H}_{\mathcal{L}} = \partial_i^\lambda L\, dy^i \wedge \omega_\lambda - H_{\mathcal{L}}\omega, \qquad H_{\mathcal{L}} = y_\lambda^j \partial_j^\lambda L - L \tag{3.4}$$

where $\Pi$ is given by (3.3).

Taking the exterior derivative of the Lagrangian density (3.1), we get the section

$$d\mathcal{L}: \quad J^1E \to \overset{m}{\wedge} T^*M \otimes V^*J^1E$$

$$d\mathcal{L} = (\partial_i L\, dy^i + \partial_i^\lambda L\, dy_\lambda^i) \otimes \omega \tag{3.5}$$

Then the *Euler–Lagrange operator* associated with the Lagrangian density $\mathcal{L}$ is the morphism over $J^1E$ defined by

$$\mathcal{E}_{\mathcal{L}} = d\mathcal{L} - \Omega_L: \quad J^1P \to \overset{m}{\wedge} T^*M \otimes V^*J^1E$$

$$\mathcal{E}_{\mathcal{L}} = (\partial_i L - p_{\lambda i}^\lambda)\omega \otimes dy^i + (\partial_i^\lambda L - p_i^\lambda)\omega \otimes dy_\lambda^i \tag{3.6}$$

where $\Omega_L$ is given by (2.5). Note that $\mathcal{E}_{\mathcal{L}}$ is an affine epimorphism over $J^1E$. Let us denote the kernel of $\mathcal{E}_{\mathcal{L}}$ by $\mathbf{L}$; it is an affine bundle over $J^1E$. A section $r: M \to P$ satisfies the Lagrange equations if it is an integral section of $\mathbf{L}$, i.e., $j^1r(x) \in \mathbf{L}$ for all $x$ (in the domain of definition of $r$). If we write locally

$$y^i \circ r = s^i, \qquad p_i^\lambda \circ r = r_i^\lambda$$

where $s^i$ and $r_i^\lambda$ are local functions on $M$, then the Lagrange equations are

$$r_i^\lambda = \partial_i^\lambda L \circ r, \qquad \partial_\lambda r_i^\lambda = \partial_i L \circ r \tag{3.7}$$

### 3.2. Hamiltonian Formalism

A *Hamiltonian form* is a scalar $m$-form on $P$ of the following type:

$$\mathcal{H}: \quad P \to \overset{m}{\wedge} T^*E, \qquad \mathcal{H} = p_i^\lambda dy^i \wedge \omega_\lambda - H\omega \tag{3.8}$$

where $H$ is a local function on $P$.

Global Hamiltonian forms always exist because they are sections of an affine bundle over $P$ (Cariñena *et al.*, 1991).

A *Hamiltonian map* is a morphism over $E$, i.e.,

$$P \xrightarrow{\phi} J^1E, \qquad (x^\lambda, y^i, y^i_\lambda) \circ \phi = (x^\lambda, y^i, \phi^i_\lambda) \tag{3.9}$$

where $\phi^i_\lambda$ are local functions on $P$. Given a Hamiltonian form, there is a Hamiltonian map canonically associated with it, i.e.,

$$\phi\colon \quad P \to J^1E, \qquad \phi^i_\lambda = \partial^i_\lambda H \tag{3.10}$$

In analogy with the Euler–Lagrange operator (3.6), the *Hamilton operator* associated with a Hamiltonian form $\mathcal{H}$ is the morphism over $P$ defined by

$$\mathcal{E}_{\mathcal{H}} = d\mathcal{H} - \Omega_H\colon \quad J^1P \to \overset{m}{\bigwedge} T^*M \otimes V^*P$$

$$\mathcal{E}_{\mathcal{H}} = (y^i_\lambda - \partial^i_\lambda H)\omega \otimes dp^\lambda_i - (p^\lambda_{\lambda i} + \partial_i H)\omega \otimes dy^i \tag{3.11}$$

where $\Omega_H$ is given by (2.6). Note that $\mathcal{E}_{\mathcal{H}}$ is an affine epimorphism over $P$. A section $r$: $M \to P$ is said to satisfy the *Hamilton equations* relative to the Hamiltonian form $\mathcal{H}$ if $\mathcal{E}_{\mathcal{H}} \circ j^1r = 0$. Locally the Hamilton equations read

$$\partial_\lambda s^i - \partial^i_\lambda H \circ r = 0, \qquad \partial_\lambda r^\lambda_i + \partial_i H \circ r = 0 \tag{3.12}$$

## 4. CONSTRAINED SYSTEMS

As is well known, in the regular case the Lagrangian and Hamiltonian descriptions are equivalent (at least locally) (Giachetta *et al.*, 1993). On the other hand, as mentioned in the Introduction, most of the physical models meet a weaker kind of regularity, which we now consider (Binz *et al.*, 1988; Zakharov, 1992; Sardanashvily and Zakharov, 1992a,b).

### 4.1. Almost Regular Lagrangian Densities

Let $\mathcal{L}$ be a Lagrangian density and let $\Pi$ be its associated Legendre map. Let $Q = \Pi(J^1E)$. Then $\mathcal{L}$ is said to be *almost regular* if:

(i) $Q \subset P$ is a (closed and regular) submanifold and $Q \to E$ is a fibered submanifold of $P \to E$.

(ii) The Legendre map $\Pi$: $J^1E \to Q$ is a submersion with connected fibers.

Let $\mathcal{L}$ be an a.r. Lagrangian density. We say that a Hamiltonian map $\phi$: $P \to J^1E$ is *associated with* $\mathcal{L}$ if

$$\Pi \circ \phi \circ j = j \tag{4.1}$$

where $j$ denotes the canonical inclusion. In other words, the restriction $\phi | Q$: $Q \to J^1E$ is a section of the submersion $\Pi$: $J^1E \to Q$.

*Remark 4.1.* For most of the physical models the fibered manifold $\Pi$: $J^1E \to Q$ has global sections. This condition guarantees the existence of global Hamiltonian maps associated with $\mathcal{L}$.

Let $\mathcal{L}$ be an a.r. Lagrangian density. We say that a Hamiltonian form is *associated with* $\mathcal{L}$ if it is of the following type:

$$\mathcal{H}_\phi = (\Theta + \mathcal{L}) \circ \phi: \quad P \to \overset{m}{\bigwedge} T^*E$$

$$H_\phi = p_i^\lambda \phi_\lambda^i - L \circ \phi \tag{4.2}$$

where $\phi$: $P \to J^1E$ is a Hamiltonian map associated with $\mathcal{L}$ and $\Theta$ is given by (2.4).

The following proposition establishes some basic facts.

*Proposition 4.2.* Let $\mathcal{L}$ be an a.r. Lagrangian density and let $\mathcal{H}_\mathcal{L}$ be its Poincaré–Cartan form. Then $\mathcal{H}_\mathcal{L}$ uniquely determines a form $\mathcal{H}_Q$: $Q \to \bigwedge^m T^*E$ such that

$$\mathcal{H}_Q \circ \Pi = \mathcal{H}_\mathcal{L} \tag{4.3}$$

Let $\mathcal{H}_\phi$ be a Hamiltonian form associated with $\mathcal{L}$ as in (4.2) and let $\Phi$ be the Hamiltonian map associated with $\mathcal{H}_\phi$ according to (3.10). Then we have

$$\Phi \circ j = \phi \circ j \tag{4.4}$$

$$j^*\mathcal{H}_\phi = \mathcal{H}_Q \tag{4.5}$$

*Proof.* For (4.3) see Binz *et al.* (1988). From (3.10) and (4.2) we see that $\Phi = \phi + \sigma_\phi$, where $\sigma_\phi$ is the map given by

$$\sigma_\phi: \quad P \to T^*M \otimes VE$$

$$\sigma_\phi = (p_j^\mu - \partial_j^\mu L \circ \phi)\partial_\lambda^i \phi_\mu^j dx^\lambda \otimes \partial_i \tag{4.6}$$

Since (4.1) implies $\sigma_\phi \circ j = 0$, the identity (4.4) is proved. Finally,

$$\mathcal{H}_Q = \mathcal{H}_Q \circ \Pi \circ \phi \circ j = \mathcal{H}_\mathcal{L} \circ \phi \circ j = \mathcal{H}_\phi \circ j$$

*Remark 4.3.* We collect together some important identities satisfied by an a.r. Lagrangian density and any associated Hamiltonian form $\mathcal{H}_\phi$. They are

$$\partial_\lambda^i H_\phi \circ j = \phi_\lambda^i \circ j \tag{4.7}$$

$$\partial_i H_\phi \circ j = -\partial_i L \circ \phi \circ j \tag{4.8}$$

The first is the local expression of (4.4). Condition (4.8) follows from (4.2) by taking into account (4.1).

As we know the Lagrange equations define an affine bundle $\mathbf{L} \to J^1E$. If $\mathcal{L}$ is a.r., we have a canonical projection $\mathbf{L} \to Q$ given by the composition of $\mathbf{L} \to J^1E$ with the Legendre map $\Pi$: $J^1E \to Q$.

Using the identities (4.7) and (4.8), we easily prove the following proposition.

*Proposition 4.4.* Let $\mathcal{L}$ be an a.r. Lagrangian density and let $\mathcal{H}_\phi$ be a Hamiltonian form associated with it as in (4.2). Then we have

$$\operatorname{Ker} \mathscr{E}_{\mathcal{H}_\phi}|Q \subset \mathbf{L} \tag{4.9}$$

Hence, if $r$: $M \to Q \subset P$ is a section compatible with the constraints which is a solution of the Hamilton equations relative to $\mathcal{H}_\phi$, i.e., $\mathscr{E}_{\mathcal{H}_\phi} \circ j^1r = 0$, then $r$ is also a solution of the Lagrange equations.

*Remark 4.5.* From (4.4) it follows that the first Hamilton equation (3.12) relative to $\mathcal{H}_\phi$ is

$$j^1s = \phi \circ r \tag{4.10}$$

where $r$: $M \to Q \subset P$ is a section compatible with the constraints and $s$: $M \to E$ is the section determined by projecting $r$ on $E$, i.e.,

$$\begin{tikzcd} P \arrow[rr] & & E \\ & M \arrow[ul, "r"] \arrow[ur, "s"'] & \end{tikzcd}$$

Therefore, working with $\mathcal{H}_\phi$, we can catch only those solutions of Lagrange equations which satisfy (4.10). However, varying $\phi$, we catch all the solutions. Indeed, let $r$: $M \to P$ be a solution of the Lagrange equations. Then $\Pi \circ j^1s = r$, where, as before, $s$: $M \to E$ is the section determined by $r$. Now there exists a Hamiltonian map associated with $\mathcal{L}$ such that (4.10) holds (at least locally) and then one can easily see that $\mathscr{E}_{\mathcal{H}_\phi} \circ j^1r = 0$.

These considerations show that, as in the case of regular systems, also for a.r. Lagrangian densities the Lagrange equations can be put into a Hamiltonian form.

As we know from (4.5), every Hamiltonian form associated with $\mathcal{L}$ is an extension of $\mathcal{H}_Q$. Now let $\mathcal{H}$: $P \to \bigwedge^m T^*E$ be any extension of $\mathcal{H}_Q$, not necessarily of the form considered so far, and let $\xi$: $Q \to J^1P|Q$ be a section. Then the equation

$$\langle \Omega_H \circ \xi - d\mathcal{H}|Q, u\rangle = 0,$$

$$\text{for each vertical field} \quad u\colon \quad Q \to VQ \tag{4.12}$$

defines an affine subbundle $\mathbf{H} \to Q$ of $J^1P|Q \to Q$. Clearly this definition does not depend on the choice of the Hamiltonian form extending $\mathcal{H}_Q$.

The affine subbundle $\mathbf{H} \to Q$ describes the *constrained Hamilton equations* associated with the a.r. Lagrangian density $\mathcal{L}$. We say that a section $r$: $M \to P$ satisfies the *constrained Hamilton equations* if it is an integral section of $\mathbf{H}$, i.e., $j^1r(x) \in \mathbf{H}$ for all $x$ (in the domain of definition of $r$).

Of course $(\operatorname{Ker} \mathcal{E}_{\mathcal{H}_\phi})|Q \subset \mathbf{H}$, for any Hamiltonian map $\phi$ associated with $\mathcal{L}$. It follows that $\mathbf{L} \subset \mathbf{H}$ (see Remark 4.5). Moreover, by their definitions, we see that both the manifolds $\mathbf{L}$ and $\mathbf{H}$ have the same dimension, namely $m + ml + m^2l$. Hence $\mathbf{L}$ is an open submanifold of $\mathbf{H}$.

Note that an integral section $r$ of $\mathbf{H}$ does not necessarily have the property that $r = \Pi \circ j^1s$, where $s$: $M \to E$ is the section determined by projecting $r$ on $E$. It is just this fact that may occur when $\mathbf{L} \neq \mathbf{H}$.

It is convenient to have a coordinate characterization of $\mathbf{H} \to Q$.

*Proposition 4.6.* Let $\mathcal{L}$ be an a.r. Lagrangian density. Let $\xi$: $Q \to J^1P|Q$ be a section and let $y^i_\lambda \circ \xi = \xi^i_\lambda$, $p^\mu_{\lambda i} \circ \xi = \xi^\mu_{\lambda i}$ be its local expression. Then the two following facts are equivalent:

(i) $\xi$ takes its values into $\mathbf{H}$, i.e., $\xi$: $Q \to \mathbf{H} \subset J^1P|Q$.
(ii) For each Hamiltonian map $\phi$ associated with $\mathcal{L}$ we have

$$(\partial^\lambda_i \Pi^\mu_j \circ \phi)(\xi^j_\mu - \phi^j_\mu) = 0 \tag{4.13}$$

$$\xi^\lambda_{\lambda i} = \partial_i L \circ \phi + (\partial_i \Pi^\mu_j \circ \phi)(\xi^j_\mu - \phi^j_\mu) = 0 \tag{4.14}$$

*Proof.* Let $u$: $Q \to VQ$ be any vertical vector field on $Q$. Let $\phi$ be any Hamiltonian map associated with $\mathcal{L}$. Since $\Pi$: $J^1E \to Q$ is a submersion, there exists a vertical field on $J^1E$, say

$$v = v^i \partial_i + v^i_\lambda \partial^\lambda_i: \quad J^1E \to VJ^1E$$

such that (at least locally) $T\Pi \circ v \circ \phi = u$, i.e.,

$$u = (v^i \circ \phi)\partial_i + [(v^i \circ \phi)(\partial_i \Pi^\mu_j \circ \phi) + (v^i_\lambda \circ \phi)(\partial^\lambda_i \Pi^\mu_j \circ \phi)]\partial^j_\mu$$

Now let $\mathcal{H}_\phi$ be the Hamiltonian form determined by $\phi$. We have

$$\Omega_H \circ \xi - d\mathcal{H}_\phi|Q = (\xi^\lambda_{\lambda i} + \partial_i H_\phi)\omega \otimes dy^i + (\partial^i_\lambda H_\phi - \xi^i_\lambda)\omega \otimes dp^\lambda_i$$

from which we get

$$\begin{aligned} \langle \Omega_H \circ \xi - d\mathcal{H}_\phi|Q, u\rangle = {} & (v^i \circ \phi)[\xi^\lambda_{\lambda i} + \partial_i H_\phi + (\partial_i \Pi^\mu_j \circ \phi)(\partial^j_\mu H_\phi - \xi^j_\mu)] \\ & + (v^i_\lambda \circ \phi)(\partial^\lambda_i \Pi^\mu_j \circ \phi)(\partial^j_\mu H_\phi - \xi^j_\mu) \end{aligned} \tag{4.15}$$

Now the result follows from the identities (4.7) and (4.8).

An interesting consequence of the proposition is the following useful characterization of $\mathbf{L}$.

*Corollary 4.7.* Let $\xi$: $Q \to \mathbf{H}$ be a section. Then $\xi$ takes its values into $\mathbf{L}$ iff it projects onto a section $\xi_Q$: $Q \to J^1E$ of $\Pi$, i.e.,

$$\begin{tikzcd} & \mathbf{H} \subset J^1P|Q \arrow[dl] & \\ J^1E \arrow[rr, "\Pi"] & & Q \arrow[ul, "\xi"'] \arrow[ll, dashed, bend left, "\xi_Q"] \end{tikzcd}$$

*Proof.* If $\xi$ is a section of $\mathbf{L} \to Q$, the first equation (3.7) shows that the property (4.16) holds. Conversely, let $\xi$: $Q \to \mathbf{H}$ be a section projecting onto a section $\xi_Q$: $Q \to J^1E$ of $\Pi$. Then taking any associated Hamiltonian map which extends $\xi_Q$, (4.14) and (3.7) show that $\xi$ takes its values into $\mathbf{L}$.

*Remark 4.8.* The relation of the pde described by $\mathbf{H}$ with the other equations in the literature is as follows. We can show that the three following facts are equivalent:

(i) A section $\sigma$: $M \to J^1E$ satisfies the *Cartan equation*, i.e.,

$$\sigma^*(u \rfloor d\mathcal{H}_{\mathcal{L}}) = 0 \qquad \text{for each vertical field} \quad u: \quad J^1E \to VJ^1E$$

(ii) The section $r = \Pi \circ \sigma$: $M \to Q \subset P$ is an integral section of $\mathbf{H}$.
(iii) $r$ satisfies the *Hamilton–De Donder equation*, i.e.,

$$r^*(u \rfloor d\mathcal{H}_Q) = 0 \qquad \text{for each vertical field} \quad u: \quad Q \to VQ$$

The relationship between $\mathbf{L}$ and $\mathbf{H}$ is connected with the 'second-order equation' problem (Bintz *et al.*, 1988; Gotay and Nester, 1980) as follows. Let $\sigma$: $M \to J^1E$ be a solution of the Cartan equation and let $s$: $M \to E$ be the section obtained by projecting on $E$. Does its lift $j^1s$ satisfy the Cartan equation? Or, equivalently, does $s$ satisfy the second-order Lagrange equations? The answer is positive iff the section $r = \Pi \circ \sigma$: $M \to P$ takes its values into $\mathbf{L}$.

## 4.2. Affine and Quadratic Lagrangian Densities

Two classes of Lagrangian densities deserve special consideration, namely affine and quadratic Lagrangian densities (in the field derivatives). A Lagrangian density is *affine* if $D\Pi = 0$ and *quadratic* if $D^2\Pi = 0$ ($D$ denotes the fiber derivative).

For instance, as is well known, the Lagrangian density of general relativity, in the Palatini formalism, is affine. On the other hand, gauge theories provide examples in which the Lagrangian densities are quadratic.

Let us see that for these two classes of Lagrangian densities we have $\mathbf{L} = \mathbf{H}$.

*Proposition 4.9.* Let $\mathcal{L}$ be an affine Lagrangian density. Then $\mathcal{L}$ is a.r. and $\mathbf{L} = \mathbf{H}$.

*Proof.* The local expression of $\mathcal{L}$ is

$$L = \Pi_i^\lambda y_\lambda^i + a$$

where $\Pi_i^\lambda$ and $a$ are local functions on $E$. Its Legendre map $\Pi$ is given by the composition

$$\begin{tikzcd} J^1E \arrow[rr, "\Pi"] \arrow[dr] & & Q \subset P \\ & E \arrow[ur, "\chi"'] & \end{tikzcd}$$

$$\chi = \Pi_i^\lambda \omega_\lambda \otimes dy^i$$

where $Q = \chi(E)$ is the image of the (global) section $\chi$. Clearly $\mathcal{L}$ is a.r. Moreover, since any morphism $Q \to J^1E$ over $E$ is a section of $\Pi$, from the Corollary 4.7 it follows that $\mathbf{L} = \mathbf{H}$.

*Proposition 4.10.* Let $\mathcal{L}$ be an a.r. quadratic Lagrangian density. Then $\mathbf{L} = \mathbf{H}$.

*Proof.* Since the Legendre map $\Pi$ is affine, $\Pi$: $J^1E \to Q$ is an affine bundle and its associated vector bundle is

$$Q \underset{E}{\times} \operatorname{Ker} D\Pi \to Q$$

where the fiber derivative

$$\begin{tikzcd} T^*M \otimes VE \arrow[rr, "D\Pi"] \arrow[dr] & & \overset{m-1}{\wedge} T^*M \otimes V^*E \arrow[dl] \\ & E & \end{tikzcd}$$

is a linear morphism of constant rank over $E$.

Let $\xi$: $Q \to \mathbf{H}$ be a section and let $\xi_Q$: $Q \to J^1E$ be the morphism over $E$ determined by $\xi$. Let $\phi$ be any associated Hamiltonian map. Then (4.13) shows that $\xi_Q - \phi | Q$ is a section of

$$Q \underset{E}{\times} \operatorname{Ker} D\Pi \to Q$$

Since we can write $\xi_Q = \phi | Q + \xi_Q - \phi | Q$, it follows that $\xi_Q$ is a section of $\Pi$: $J^1E \to Q$. Hence Corollary 4.7 implies that $\xi$ is a section of $\mathbf{L} \to \mathbf{Q}$.

*Remark 4.11.* The Lagrangian of the free relativistic particle (Hanson *et al.*, 1976) provides an example in which $\mathbf{L} \neq \mathbf{H}$. Here the fibers of $\mathbf{H} \to Q$ are real lines, while those of $\mathbf{L} \to Q$ are the corresponding positive half-lines.

## 5. YANG–MILLS THEORY

The Yang–Mills Lagrangian density, which is quadratic and a.r., provides a good example to illustrate the previous theory.

### 5.1. The Configuration and Momentum Bundles

Let $\mathcal{P} \to M$ be a principal fiber bundle with structure Lie group $G$ (Kobayashi and Nomizu, 1963) and let us denote by $C \to M$ the bundle of its principal connections (Mangiarotti and Modugno, 1985). As is well known, this is an affine bundle whose associated vector bundle is

$$T^*M \underset{M}{\otimes} V_G\mathcal{P} \to M$$

Here the quotient bundle $V_G\mathcal{P} = V\mathcal{P}/G \to M$ is the vector bundle of right invariant vertical vector fields on $\mathcal{P}$ (*gauge algebra bundle*). Coordinates on $C$ and $J^1C$ are denoted by $(x^\lambda, a^r_\mu)$ and $(x^\lambda, a^r_\mu, a^r_{\lambda,\mu})$, respectively.

The affine bundle $C \to M$ is the configuration bundle of the (free) Yang–Mills theory.

Let $A: M \to C$ be a section, i.e., a principal connection (*gauge potential*). Locally we write

$$(x^\lambda, a^r_\mu) \circ A = (x^\lambda, A^r_\mu) \tag{5.1}$$

where $A^r_\mu$ are local functions on $M$. The curvature of $A$ (*field strength*) is the following $V_G\mathcal{P}$-valued 2-form on $M$:

$$F_A: M \to \overset{2}{\bigwedge} T^*M \otimes V_G\mathcal{P}$$

$$F_A = \frac{1}{2}(\partial_\lambda A^r_\mu - \partial_\mu A^r_\lambda + c^r_{pq}A^p_\lambda A^q_\mu)dx^\lambda \wedge dx^\mu \otimes e_r \tag{5.2}$$

where $(e_r)$ is a local basis for the sections of $V_G\mathcal{P} \to M$, while the Lie bracket $[e_p, e_q] = c^r_{pq}e_r$ defines the right structure constants of $G$.

Later we will need the fact that a principal connection $A$ induces a covariant derivative $\nabla^A$ in the vector bundle $V_G\mathcal{P} \to M$, whose parameters are determined by the following equation:

$$\partial_\lambda \rfloor \nabla^A e_q = c^r_{pq}A^p_\lambda e_r \tag{5.3}$$

The basic property of the configuration bundle $C \to M$ is that its first-order jet space $J^1C$ admits the following canonical splitting over $C$ (Mangiarotti and Modugno, 1985):

$$J^1C = \Sigma \underset{C}{\oplus} \overset{2}{\wedge} T^*M \otimes V_G\mathcal{P}$$

$$a^r_{\lambda,\mu} = \frac{1}{2}\,(a^r_{\lambda,\mu} + a^r_{\mu,\lambda} - c^r_{pq}a^p_\lambda a^q_\mu) + \frac{1}{2}\,(a^r_{\lambda,\mu} - a^r_{\mu,\lambda} + c^r_{pq}a^p_\lambda a^q_\mu) \qquad (5.4)$$

where $\Sigma \to C$ is an affine subbundle of $J^1C \to C$ whose associated vector bundle is $C \times_M \vee_2 T^*M \otimes V_G\mathcal{P} \to M$.

The sections $S$: $C \to \Sigma$ give rise to distinguished connections on the configuration bundle $C \to M$ which, locally, are characterized by

$$a^r_{\lambda,\mu} \circ S = S^r_{\lambda,\mu}, \qquad S^r_{\lambda,\mu} - S^r_{\mu,\lambda} + c^r_{pq}a^p_\lambda a^q_\mu = 0 \qquad (5.5)$$

as follows from (5.4). These connections will be used to construct associated Hamiltonian maps.

We have the projection

$$F\colon \quad J^1C \to \overset{2}{\wedge} T^*M \otimes V_G\mathcal{P}$$

$$F = \frac{1}{4}\,F^r_{\lambda\mu}dx^\lambda \wedge dx^\mu \otimes e_r, \qquad F^r_{\lambda\mu} = a^r_{\lambda,\mu} - a^r_{\mu,\lambda} + c^r_{pq}a^p_\lambda a^q_\mu \qquad (5.6)$$

which has the property that

$$F \circ j^1A = \frac{1}{2}\,F_A \qquad (5.7)$$

Let us see that also the momentum bundle $P \to C \to M$ admits a canonical splitting over $C$. Induced coordinates on $P$ are denoted by $(x^\lambda, a^r_\mu, p^{\lambda,\mu}_r)$. We recall that $P = C \times_M \wedge^{m-1} T^*M \otimes TM \otimes V^*_GP$, since $V^*C = C \times_M TM \otimes V^*_GP$. Let us introduce the bundles

$$P^- = \overset{m}{\wedge} T^*M \otimes \overset{2}{\wedge} TM \otimes V^*_G\mathcal{P} = \overset{m-2}{\wedge} T^*M \otimes V^*_G\mathcal{P}$$

$$P^+ = \overset{m}{\wedge} T^*M \otimes \underset{2}{\vee} TM \otimes V^*_G\mathcal{P} \qquad (5.8)$$

whose induced coordinates are denoted by $(x^\lambda, p^{[\lambda,\mu]}_r)$ and $(x^\lambda, p^{(\lambda,\mu)}_r)$, respectively, and the pullback bundles

$$Q = C \underset{M}{\times} P^{-}, \qquad R = C \underset{M}{\times} P^{+} \tag{5.9}$$

It follows that the momentum manifold $P$ can be written as follows:

$$\begin{aligned} P &= Q \underset{C}{\bigoplus} R \\ p_r^{\lambda,\mu} &= p_r^{[\lambda,\mu]} + p_r^{(\lambda,\mu)} \\ p_r^{[\lambda,\mu]} &= \frac{1}{2}\,(p_r^{\lambda,\mu} - p_r^{\mu,\lambda}) \\ p_r^{(\lambda,\mu)} &= \frac{1}{2}\,(p_r^{\lambda,\mu} + p_r^{\mu,\lambda}) \end{aligned} \tag{5.10}$$

The canonical splittings (5.4) and (5.10) play a key role in the Yang–Mills theory.

## 5.2. The Yang–Mills Equations

Suppose that $M$ is an oriented manifold that carries a pseudo-Riemannian metric $g$. Moreover, let $h$ be a metric on the Lie algebra $LG$ of $G$ such that the adjoint representation is orthogonal. Then the Yang–Mills Lagrangian density is

$$\begin{aligned} &\mathcal{L}:\ J^1C \to \overset{m}{\bigwedge} T^*M, \qquad \mathcal{L} = L\omega \\ &L = \frac{1}{4}\sqrt{|g|}F^r_{\lambda\mu}F^{\lambda\mu}_r, \qquad F^{\lambda\mu}_r = h_{rs}g^{\lambda\alpha}g^{\mu\beta}F^s_{\alpha\beta} \end{aligned} \tag{5.11}$$

As (5.6) shows, this is a quadratic Lagrangian density in the field derivatives. Its main property is expressed by the Legendre map $\Pi$: $J^1C \to P$,

$$p_r^{[\lambda,\mu]} \circ \Pi \equiv \Pi_r^{[\lambda,\mu]} = \sqrt{|g|}F^{\lambda\mu}_r, \qquad p_r^{(\lambda,\mu)} \circ \Pi = 0 \tag{5.12}$$

or, equivalently,

$$\frac{1}{2}\,(* \circ \Pi) = F \tag{5.13}$$

where

$$\begin{aligned} &Q \xrightarrow{*} C \underset{M}{\times} \overset{2}{\bigwedge} T^*M \otimes V_G\mathcal{P} \\ &(x^\lambda, a^r_\mu, p_r^{[\lambda,\mu]}) \mapsto \left(x^\lambda, a^r_\mu, \frac{1}{\sqrt{|g|}}\,h^{rs}g_{\lambda\alpha}g_{\mu\beta}p_s^{[\alpha,\beta]}\right) \end{aligned} \tag{5.14}$$

is the Hodge operator. It follows that $\mathcal{L}$ is a.r.

In the sequel we will need the following partial derivative:

$$\partial_r^\mu L = -\sqrt{|g|}\, c^s_{rq} a^q_\lambda F^{\lambda\mu}_s = -c^s_{rq} a^q_\lambda \Pi^{\lambda\mu}_s \tag{5.15}$$

Taking the composition of the projection $\pi_Q$: $P \to Q$ with (5.14) and the injection $C \times_M \wedge^2 T^*M \otimes V_G P \hookrightarrow T^*M \otimes VC$, we get the following canonical soldering form on $P$:

$$P \xrightarrow{\sigma} T^*M \otimes VC$$

$$\overline{a}^r_{\lambda,\mu} \circ \sigma \equiv \sigma^r_{\lambda,\mu} = \frac{1}{\sqrt{|g|}} h^{rs} g_{\lambda\alpha} g_{\mu\beta} p_s^{[\alpha,\beta]} \tag{5.16}$$

which is linear over $C$.

Using this soldering form, we see that any connection $S$ as in (5.5) determines the following Hamiltonian map associated with $\mathcal{L}$:

$$\phi_S = S + \frac{1}{2}\sigma : P \to J^1 C$$

$$(\phi_S)^r_{\lambda,\mu} = S^r_{\lambda,\mu} + \frac{1}{2}\sigma^r_{\lambda,\mu} \tag{5.17}$$

Indeed (5.5) and (5.13) show that $\Pi \circ \phi_S = \pi_Q$: $P \to Q$ and hence the condition (4.1) is satisfied. Note that we have

$$\sigma_{\phi_S} = 0 \tag{5.18}$$

everywhere on $P$ [see (4.6)].

As we know from the previous section, we have $\mathbf{L} = \mathbf{H}$ and hence (3.7) yields

$$\mathbf{H} \equiv \{x^\lambda, a^r_\mu, p_r^{(\lambda,\mu)} = 0, p_r^{[\lambda,\mu]} = \sqrt{|g|} F_r^{\lambda\mu}, p^{\lambda,\mu}_{\lambda r} = -c^s_{rq} a^q_\lambda p_s^{[\lambda,\mu]}\} \tag{5.19}$$

where we have used (5.12) and (5.15). Of course, the integral sections $r$: $M \to P$ of $\mathbf{H}$ are the solutions of the Yang–Mills equations, i.e.,

$$* \circ \chi = F_A, \qquad \nabla^A \chi = 0 \tag{5.20}$$

where $A$ is a principal connection and $\chi$ is a section of $P^- \to M$.

*Remark 5.1.* Going back for a moment to the general theory and recalling (2.7), let us introduce the following vector bundle over $Q$:

$$\begin{tikzcd} Z = \mathrm{Ker}\,\bar{\Omega}_H|Q \arrow[rr, hook] \arrow[dr] & & T^*M \otimes VP|Q \arrow[dl] \\ & Q & \end{tikzcd}$$

$$Z \equiv \{x^\lambda, y^i, p_i^\lambda, \bar{y}_\lambda^i = 0, \bar{p}_{\lambda i}^\lambda = 0\}, \qquad (x^\lambda, y^i, p_i^\lambda) \in Q \tag{5.21}$$

From (4.12) one can easily see that $Z \subset \bar{\mathbf{H}}$. Moreover, the Hamilton equations [see (3.12) and (4.14)] have a freedom up to the trace condition $\bar{p}_{\lambda i}^\lambda = 0$. It follows that they are more appropriately described by the quotient bundle $\mathbf{H}/Z \to Q$.

In our present situation it can be shown that the further relation

$$\mathbf{H}/Z \subset J^1Q/(Z \cap T^*M \otimes VQ)$$

holds. This means that, up to $Z$, the Hamiltonian constraint $Q \subset P$ is a *first-class* constraint (extending the terminology of mechanics) (Śniatycki, 1974).

## 5.3. The Complete Family of Hamiltonian Forms

We know that any connection $S$: $C \to \Sigma \subset J^1C$ determines a Hamiltonian map $\phi_S$: $P \to J^1C$ associated with the Yang–Mills Lagrangian density. According to (4.2), the corresponding Hamiltonian form $\mathcal{H}_{\phi_S} \equiv \mathcal{H}_S$ is

$$\mathcal{H}_S\colon \quad P \to \bigwedge^m T^*C, \qquad \mathcal{H}_S = p_r^{\lambda,\mu} da_\mu^r \wedge \omega_\lambda - H_S\omega$$

$$H_S = p_r^{\lambda,\mu} S_{\lambda,\mu}^r + \frac{1}{4}\frac{1}{\sqrt{|g|}} h^{rs} g_{\lambda\alpha} g_{\mu\beta} p_r^{[\lambda,\mu]} p_s^{[\alpha,\beta]} \tag{5.22}$$

where we have used (5.11) and (5.17).

Note that $\mathcal{H}_S$ is quadratic in the momenta. Let $\Phi_S$ be the Hamiltonian map associated with $\mathcal{H}_S$ according to (3.10). From (5.18) it follows that $\Phi_S = \phi_S$ everywhere on $P$ and not only on $Q \subset P$. It turns out that these two properties characterize the family $\mathcal{H}_S$. Indeed, the following proposition can be easily proved.

*Proposition* 5.2. Let $\mathcal{H}$: $P \to \bigwedge^m T^*C$ be a Hamiltonian form and let $\phi$: $P \to J^1C$ be the Hamiltonian map associated with $\mathcal{H}$ according to (3.10). Let $\sigma_\phi$: $P \to T^*M \otimes VE$ be the soldering form on $P$ determined by $\phi$ as in (4.6). Suppose that:

(i) $\Pi \circ \phi \circ j = j$.
(ii) $\sigma_\phi = 0$ everywhere on $P$.
(iii) $\mathcal{H}$ is quadratic in the momenta.

Then there exists a connection $S$: $C \to \Sigma \subset J^1C$ such that $\mathcal{H} = \mathcal{H}_S$.

Let us consider the Hamilton equations relative to any Hamiltonian form $\mathcal{H}_S$. We begin with the first equation (4.10). If $r = (A, \chi)$: $M \to Q$ is a section, we have

$$\pi_\Sigma \circ j^1A + F \circ j^1A = j^1A = \phi_S \circ r = S \circ A + \frac{1}{2}(* \circ \chi)$$

where we have used (5.4), (5.16), and (5.17). Here $\pi_\Sigma$ denotes the canonical projection $J^1C \to \Sigma$. Recalling (5.7), we see that the section $r$ satisfies the first Hamilton equation relative to $\mathcal{H}_S$ iff we have

$$S \circ A = \pi_\Sigma \circ j^1A \tag{5.23}$$

$$* \circ \chi = F_A \tag{5.24}$$

Note that the curvature $F_A$ is zero iff $A$ is an integral section of the connection $S$. The condition (5.23) plays the role of a *gauge-type* condition. The second Hamilton equation (3.12) reproduces the equation $\nabla^A\chi = 0$.

An interesting fact is that the family (5.22) is *complete* in the sense that any solution of the Yang–Mills equations can be seen as a solution of the Hamilton equations relative to a certain Hamiltonian form $\mathcal{H}_S$ (see also Remark 4.5).

Indeed let $A$: $M \to C$ be a principal connection. Let $K$ be a symmetric linear connection on $M$ (for instance, the Levi-Civita connection of the metric $g$ on $M$). Then we can define a linear connection on the vector bundle $T^*M \otimes V_G\mathcal{P} \to M$ and hence on $C \to M$. Using the projection $\pi_\Sigma$: $J^1C \to \Sigma$, we get the following connection:

$$S\colon \quad C \to \Sigma \subset J^1C$$

$$\begin{aligned} S^r_{\lambda,\mu} = \frac{1}{2}\{&\partial_\lambda A^r_\mu + \partial_\mu A^r_\lambda - c^r_{pq}(A^p_\lambda a^q_\mu + A^p_\mu a^q_\lambda) \\ &- 2K^\nu{}_{\lambda\mu}(a^r_\nu - A^r_\nu) - c^r_{pq}a^p_\lambda a^q_\mu\}\end{aligned} \tag{5.25}$$

A direct check using (5.25) shows that the gauge condition (5.23) is satisfied. Hence if $r = (A, \chi)$ is a solution of the Yang–Mills equations, $S$ is the connection (5.25) associated with $A$, and $\mathcal{H}_S$ is the corresponding Hamiltonian form, then $r$ is also a solution of the Hamilton equations relative to $\mathcal{H}_S$.

*Remark 5.3.* Note that (5.23) is a more general condition than the usual gauge conditions in gauge theory. As is well known, these latter single out (up to the Gribov ambiguity) a representative within a given gauge equivalence class. On the other hand, there are solutions of the Yang–Mills equations which are not singled out by the gauge conditions known in gauge theory. In this sense, these latter do not form a complete family.

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